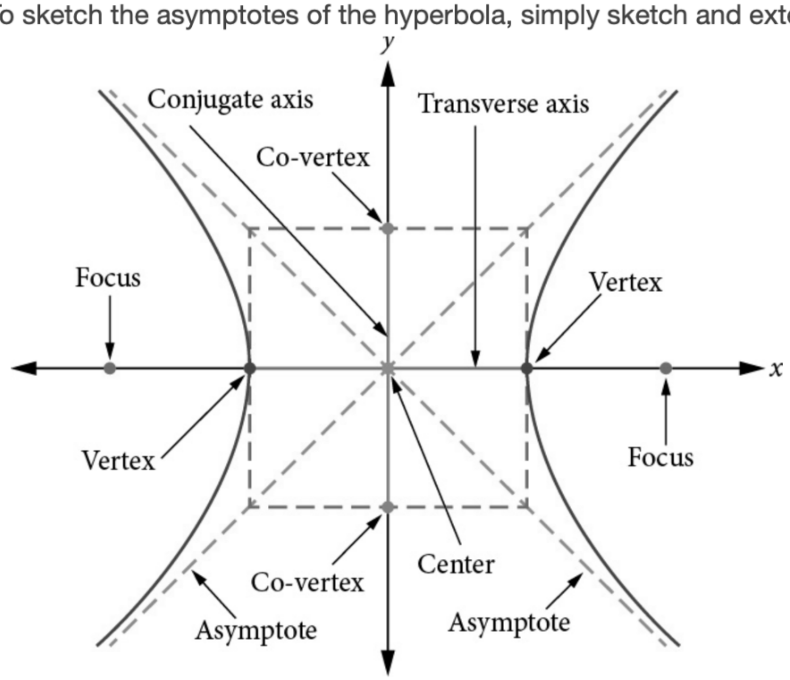
As with the ellipse, every hyperbola has two axes of symmetry. The **transverse axis** is a line segment that passes through the center of the hyperbola and has vertices as its endpoints. The foci lie on the line that contains the transverse axis. The **conjugate axis** is perpendicular to the transverse axis and has the co-vertices as its endpoints. The **center of a hyperbola** is the midpoint of both the transverse and conjugate axes, where they intersect. Every hyperbola also has two **asymptotes** that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes. The **central rectangle** of the hyperbola is centered at the origin with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle.



Deriving the Equation of a Hyperbola Centered at the Origin

Let $(-c, 0)$ and $(c, 0)$ be the foci of a hyperbola centered at the origin. The hyperbola is the set of all points (x, y) such that the difference of the distances from (x, y) to the foci is constant. See Figure 4.

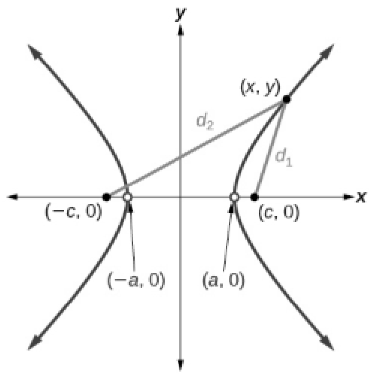


Figure 4

If $(a, 0)$ is a vertex of the hyperbola, the distance from $(-c, 0)$ to $(a, 0)$ is $a-(-c)=a+c$. The distance from $(c, 0)$ to $(a, 0)$ is $c-a$. The difference of the distances from the foci to the vertex is

$$(a+c)-(c-a)=2a$$

If (x, y) is a point on the hyperbola, we can define the following variables:

$$\begin{aligned} d_2 &= \text{the distance from } (-c, 0) \quad \text{to } (x, y) \\ d_1 &= \text{the distance from } (c, 0) \quad \text{to } (x, y) \end{aligned}$$

By definition of a hyperbola, d_2-d_1 is constant for any point (x, y) on the hyperbola. We know that the difference of these distances is $2a$ for the vertex $(a, 0)$. It follows that $d_2-d_1=2a$ for any point on the hyperbola. As with the derivation of the equation of an ellipse, we will begin by applying the distance formula. The rest of the derivation is algebraic. Compare this derivation with the one from the previous section for ellipses.

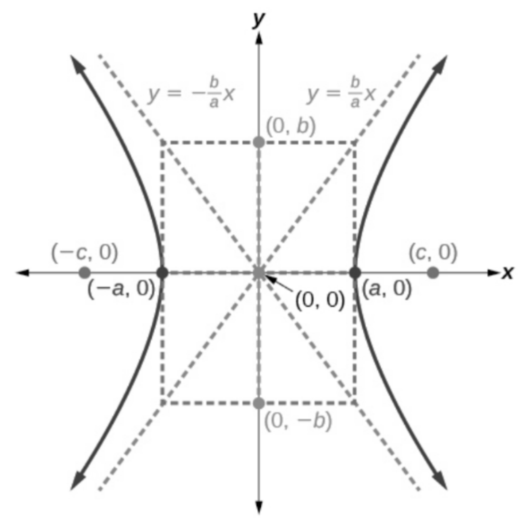
$$\begin{array}{ll}
d_2-d_1=\sqrt{(x-(-c))^2+(y-0)^2}-\sqrt{(x-c)^2+(y-0)^2}=2a & \text{Distance Formula} \\
\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a & \text{Simplify expressions.} \\
\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2} & \text{Move radical to opposite side.} \\
(x+c)^2+y^2=\left(2a+\sqrt{(x-c)^2+y^2}\right)^2 & \text{Square both sides.} \\
x^2+2cx+c^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 & \text{Expand the squares.} \\
x^2+2cx+c^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2 & \text{Expand remaining square.} \\
2cx=4a^2+4a\sqrt{(x-c)^2+y^2}-2cx & \text{Combine like terms.} \\
4cx-4a^2=4a\sqrt{(x-c)^2+y^2} & \text{Isolate the radical.} \\
cx-a^2=a\sqrt{(x-c)^2+y^2} & \text{Divide by 4.} \\
\left(cx-a^2\right)^2=a^2\left(\sqrt{(x-c)^2+y^2}\right)^2 & \text{Square both sides.} \\
c^2x^2-2a^2cx+a^4=a^2\left(x^2-2cx+c^2+y^2\right) & \text{Expand the squares.} \\
c^2x^2-2a^2cx+a^4=a^2x^2-2a^2cx+a^2c^2+a^2y^2 & \text{Distribute } a^2. \\
a^4+c^2x^2=a^2x^2+a^2c^2+a^2y^2 & \text{Combine like terms.} \\
c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4 & \text{Rearrange terms.} \\
x^2\left(c^2-a^2\right)-a^2y^2=a^2\left(c^2-a^2\right) & \text{Factor common terms.} \\
x^2b^2-a^2y^2=a^2b^2 & \text{Set } b^2=c^2-a^2. \\
\frac{x^2b^2}{a^2b^2}-\frac{a^2y^2}{a^2b^2}=\frac{a^2b^2}{a^2b^2} & \text{Divide both sides by } a^2b^2 \\
\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 &
\end{array}$$

This equation defines a hyperbola centered at the origin with vertices $(\pm a, 0)$ and co-vertices $(0 \pm b)$.

STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER (0,0)

The standard form of the equation of a hyperbola with center $(0, 0)$ and transverse axis on the *x*-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



where

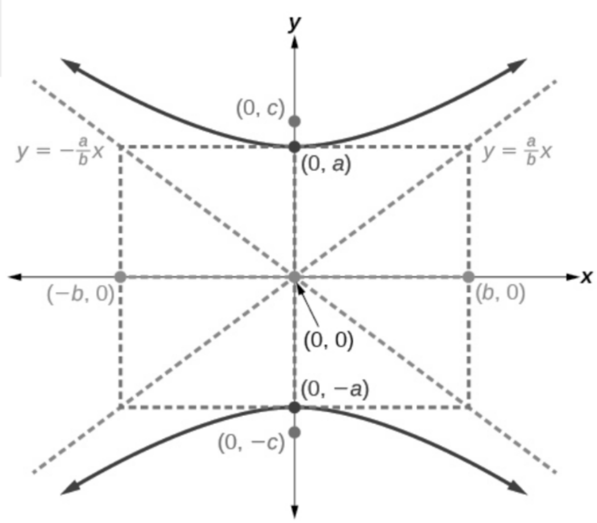
- the length of the major transverse axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the minor conjugate axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the distance between the foci is $2c$, where $\boxed{c^2 = a^2 + b^2}$
- the coordinates of the foci are $(\pm c, 0)$
- the equations of the asymptotes are $y = \pm \frac{b}{a}x$

The standard form of the equation of a hyperbola with center $(0, 0)$ and transverse axis on the *y*-axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(\pm b, 0)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y = \pm \frac{a}{b} x$



HOW TO

Given the equation of a hyperbola in standard form, locate its vertices and foci.

1. Determine whether the transverse axis lies on the *x*- or *y*-axis. Notice that a^2 is always under the variable with the positive coefficient. So, if you set the other variable equal to zero, you can easily find the intercepts. In the case where the hyperbola is centered at the origin, the intercepts coincide with the vertices.
   a. If the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the transverse axis lies on the *x*-axis. The vertices are located at $(\pm a, 0)$, and the foci are located at $(\pm c, 0)$.
   b. If the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then the transverse axis lies on the *y*-axis. The vertices are located at $(0, \pm a)$, and the foci are located at $(0, \pm c)$.
2. Solve for a using the equation $a = \sqrt{a^2}$.
3. Solve for c using the equation $c = \sqrt{a^2 + b^2}$.

Locating a Hyperbola’s Vertices and Foci

Identify the vertices and foci of the hyperbola with equation $\frac{y^2}{49} - \frac{x^2}{32} = 1.$

$a^2 = 49 \quad b^2 = 32$

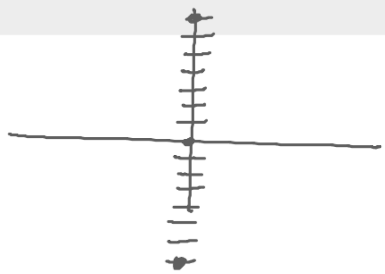
$a = \pm 7$

$c^2 = a^2 + b^2$

$c^2 = 49 + 32$

$c^2 = 81$

$c = \pm 9$



Vertices

$(0,7)\ (0,-7)$

$(0,\pm 7)$

Foci:

$(0,9)\ (0,-9)$

$(0,\pm 9)$

Identify the vertices and foci of the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{25} = 1.$

$a^2 = 9$

$a = \pm 3$

$c^2 = a^2 + b^2$

$9 + 25$

34

$c = \pm\sqrt{34}$

Vertices

$(3,0)$ $(-3,0)$

$(\pm 3,0)$

Foci

$(\sqrt{34},0)$ $(-\sqrt{34},0)$

$(\pm\sqrt{34},0)$

HOW TO

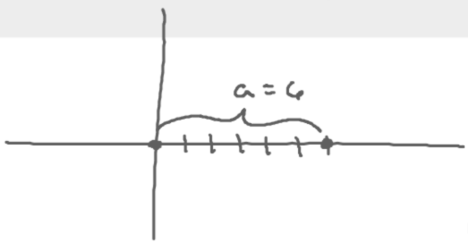
Given the vertices and foci of a hyperbola centered at $(0, \quad 0)$, write its equation in standard form.

1. Determine whether the transverse axis lies on the *x*- or *y*-axis.
   a. If the given coordinates of the vertices and foci have the form $(\pm a, 0)$ and $(\pm c, 0)$, respectively, then the transverse axis is the *x*-axis. Use the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$
   b. If the given coordinates of the vertices and foci have the form $(0, \pm\, a)$ and $(0, \pm\, c)$, respectively, then the transverse axis is the *y*-axis. Use the standard form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$
2. Find b^2 using the equation $b^2 = c^2 - a^2$.
3. Substitute the values for a^2 and b^2 into the standard form of the equation determined in Step 1.

Finding the Equation of a Hyperbola Centered at (0,0) Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices $(\pm 6, 0)$ and foci $\left(\pm 2\sqrt{10}, 0\right)$?

$c = 2\sqrt{10}$



$$\frac{x^2}{36} - \frac{y^2}{b^2} = 1$$

$$c^2 = a^2 + b^2$$

$$(2\sqrt{10})^2 = 6^2 + b^2$$

$$40 = 36 + b^2$$

$$b^2 = 4$$

$$\frac{x^2}{36} - \frac{y^2}{4} = 1$$

What is the standard form equation of the hyperbola that has vertices $(0, \pm 2)$ and foci $\left(0, \pm 2\sqrt{5}\right)$?

$$\frac{y^2}{4} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{4} - \frac{x^2}{16} = 1$$

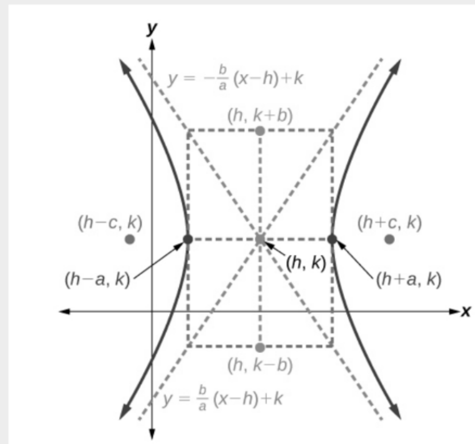
$$c^2 = a^2 + b^2$$

$$(2\sqrt{5})^2 = 2^2 + b^2$$

$$20 = 4 + b^2$$

$$b^2 = 16$$

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the *x*-axis is



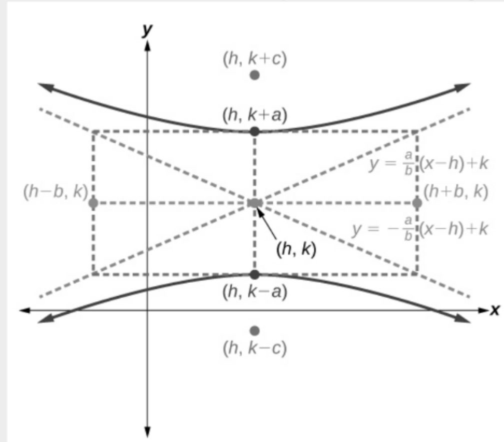
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h \pm c, k)$

The asymptotes of the hyperbola coincide with the diagonals of the central rectangle. The length of the rectangle is $2a$ and its width is $2b$. The slopes of the diagonals are $\pm\frac{b}{a}$, and each diagonal passes through the center (h, k) . Using the **point-slope formula**, it is simple to show that the equations of the asymptotes are $y = \pm\frac{b}{a}(x-h) + k$. See Figure 7**a**

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the y-axis is



$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h, k \pm c)$

Using the reasoning above, the equations of the asymptotes are $y = \pm\frac{a}{b}(x-h) + k$. See Figure 7**b**.

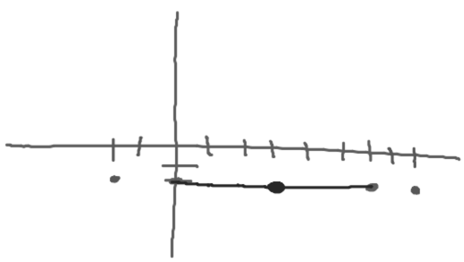
HOW TO

Given the vertices and foci of a hyperbola centered at (h, k), write its equation in standard form.

1. Determine whether the transverse axis is parallel to the *x*- or *y*-axis.
   a. If the *y*-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the *x*-axis. Use the standard form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$
   b. If the *x*-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the *y*-axis. Use the standard form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$
2. Identify the center of the hyperbola, (h, k), using the midpoint formula and the given coordinates for the vertices.
3. Find a^2 by solving for the length of the transverse axis, $2a$, which is the distance between the given vertices.
4. Find c^2 using h and k found in Step 2 along with the given coordinates for the foci.
5. Solve for b^2 using the equation $b^2 = c^2 - a^2$.
6. Substitute the values for h, k, a^2, and b^2 into the standard form of the equation determined in Step 1.

Finding the Equation of a Hyperbola Centered at (*h*, *k*) Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices at $(0, -2)$ and $(6, -2)$ and foci at $(-2, -2)$ and $(8, -2)$?



$2a = 6$

$a = 3$

$c = 5$

$25 = 9 + b^2$

$b^2 = 16$

$(\underset{h}{3}, \underset{k}{-2}) \rightarrow$ Center

$$\frac{(x-3)^2}{9} - \frac{(y+2)^2}{16} = 1$$

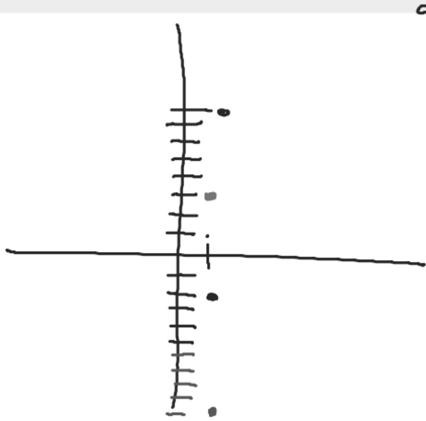
What is the standard form equation of the hyperbola that has vertices $(1, -2)$ and $(1, 8)$ and foci $(1, -10)$ and $(1, 16)$?

$2a = 10$

$a = 5$

$c = 13$

Center $(1, 3)$



$c^2 = a^2 + b^2$

$169 = 25 + b^2$

$b^2 = 144$

$$\frac{(y-3)^2}{25} - \frac{(x-1)^2}{144} = 1$$

HOW TO

Given a standard form equation for a hyperbola centered at $(0, 0)$, sketch the graph.

1. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the vertices, co-vertices, and foci; and the equations for the asymptotes.

   a. If the equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$ then

      - the transverse axis is on the *x*-axis
      - the coordinates of the vertices are $(\pm a, 0)$
      - the coordinates of the co-vertices are $(0, \pm\, b)$
      - the coordinates of the foci are $(\pm c, 0)$
      - the equations of the asymptotes are $y = \pm\frac{b}{a}x$

   b. If the equation is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then

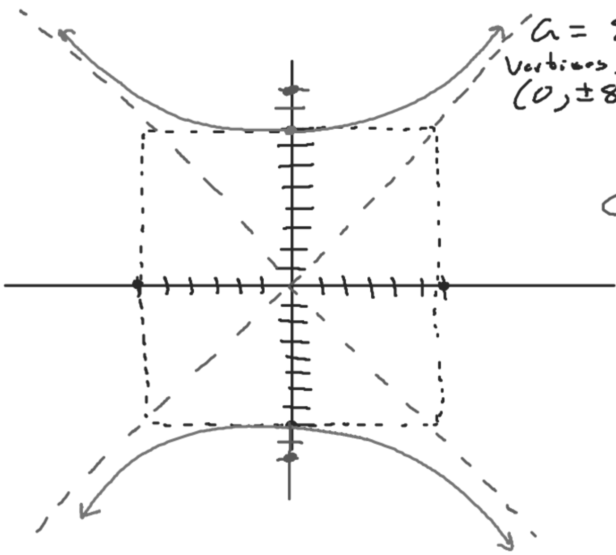
      - the transverse axis is on the *y*-axis
      - the coordinates of the vertices are $(0, \pm\, a)$
      - the coordinates of the co-vertices are $(\pm b, 0)$
      - the coordinates of the foci are $(0, \pm\, c)$
      - the equations of the asymptotes are $y = \pm\frac{a}{b}x$

3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2 + b^2}$.
4. Plot the vertices, co-vertices, foci, and asymptotes in the coordinate plane, and draw a smooth curve to form the hyperbola.

Graphing a Hyperbola Centered at (0, 0) Given an Equation in Standard Form

Graph the hyperbola given by the equation $\frac{y^2}{64} - \frac{x^2}{36} = 1.$ Identify and label the vertices, co-vertices, foci, and asymptotes.

Center $(0,0)$



$a = 8$

Vertices

$(0, \pm 8)$

$b = 6$

Co-vertices

$(\pm 6, 0)$

$c^2 = 64 + 36$

$c^2 = 100$

$c = 10$

Foci $(0, \pm 10)$

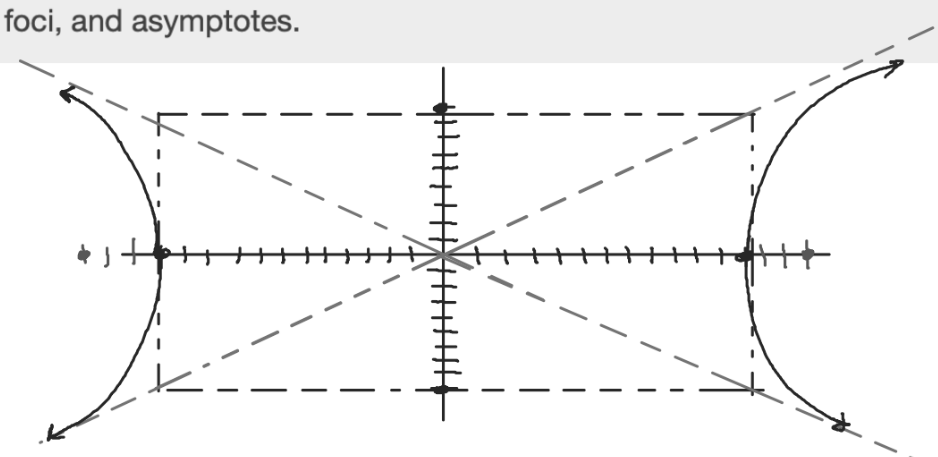
Asymptotes

$y = \pm \frac{a}{b} x$

$y = \pm \frac{8}{6} x$

$\pm \frac{4}{3} x$

Graph the hyperbola given by the equation $\frac{x^2}{144} - \frac{y^2}{81} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.



Center $(0,0)$

Vertices $(\pm 12, 0)$

$a = 12$

$b = 9$

Co-Vertices $(0, \pm 9)$

$c^2 = 144 + 81$

$c^2 = 225$

$= 15$

Foci $(\pm 15, 0)$

Asymptotes

$y \pm \frac{9}{12}x = \frac{3}{4}x$

HOW TO

Given a general form for a hyperbola centered at (h, k), sketch the graph.

1. Convert the general form to that standard form. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the center, vertices, co-vertices, foci; and equations for the asymptotes.

   a. If the equation is in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, then

      - the transverse axis is parallel to the *x*-axis
      - the center is (h, k)
      - the coordinates of the vertices are $(h \pm a, k)$
      - the coordinates of the co-vertices are $(h, k \pm b)$
      - the coordinates of the foci are $(h \pm c, k)$
      - the equations of the asymptotes are $y = \pm\frac{b}{a}(x - h) + k$

   b. If the equation is in the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, then

      - the transverse axis is parallel to the *y*-axis
      - the center is (h, k)
      - the coordinates of the vertices are $(h, k \pm a)$
      - the coordinates of the co-vertices are $(h \pm b, k)$
      - the coordinates of the foci are $(h, k \pm c)$
      - the equations of the asymptotes are $y = \boxed{\pm\frac{a}{b}(x - h) + k}$

3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2 + b^2}$.
4. Plot the center, vertices, co-vertices, foci, and asymptotes in the coordinate plane and draw a smooth curve to form the hyperbola.

Graph the hyperbola given by the standard form of an equation $\frac{(y+4)^2}{100} - \frac{(x-3)^2}{64} = 1$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Center $(3,-4)$

$a^2 = 100$ $\quad b^2 = 64$

$a = 10$ $\quad b = 8$

Vertices

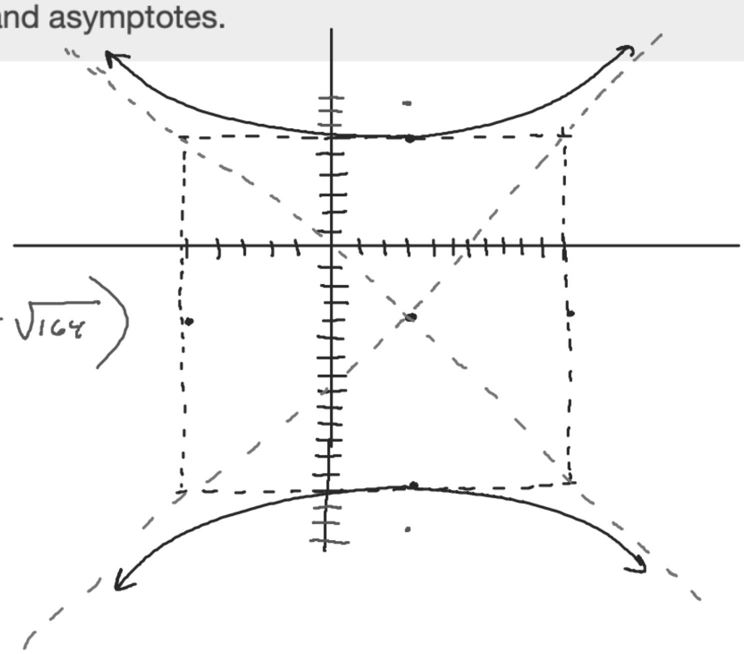
$(3,6)$ $(3,-14)$

Co-Vertices

$(11,-4)$ $(-5,-4)$

Foci

$(3, -4 \pm \sqrt{164})$



Asymptotes

$y = \pm \frac{a}{b}(x-h)+k$

$y = \pm \frac{10}{8}(x-3)-4$

$\pm \frac{5}{4}(x-3)-4$

$c^2 = 100+64$

$c^2 = 164$

$c = \pm\sqrt{164}$

Graphing a Hyperbola Centered at (*h*, *k*) Given an Equation in General Form

Graph the hyperbola given by the equation $9x^2 - 4y^2 - 36x - 40y - 388 = 0$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.